Lemniscate Trees of Random Polynomials and Asymptotic Enumeration of Morse Functions on the 2-Sphere

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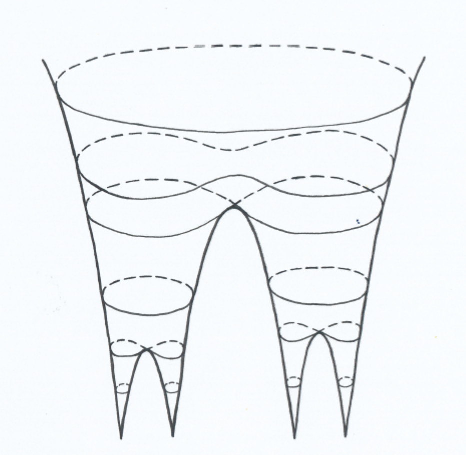
Outline

- Lemniscate trees of random polynomials
- Asymptotic enumeration of Morse functions on the 2-sphere

Introduction



Problem: Understand the "landscape" of level sets of the modulus of a polynomial.

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- If P is lemniscate generic it has n singular lemniscates, each with one singular connected component passing through a critical point

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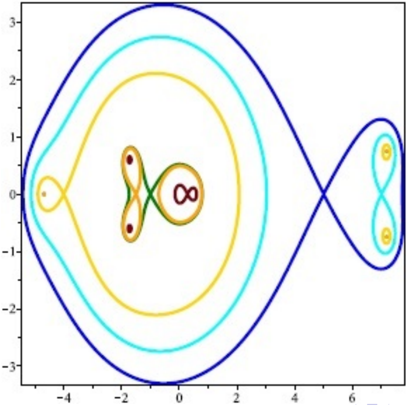
Example

Figure: $f(z) = \frac{1}{7}z^7 - \frac{27}{28}z^6 - \frac{419}{70}z^5 + \frac{209}{8}z^4 + \frac{1415}{14}z^3 + \frac{809}{14}z^2 - 60z$



Connection to Combinatorics

Catanese and Paluszny show that generic lemniscate configurations of degree $n+1$ polynomials are in one-to-one correspondence with the collection $\mathcal{T}_n$ of "lemniscate trees" on n vertices, i.e. labeled, rooted trees in which:

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1. vertices have at most 2 children
2. vertex labels increase along any path directed away from the root

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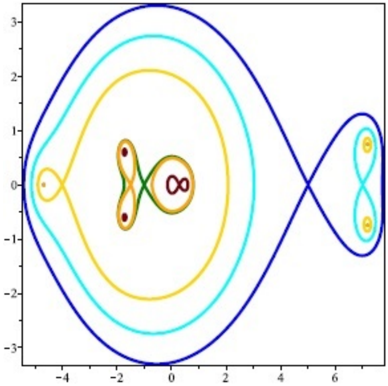

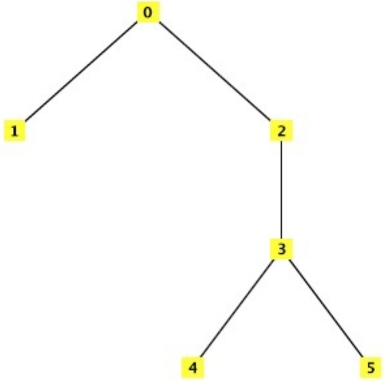

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- Questions:
  1. What is the distribution of nodes with 2 children among trees in $\mathcal{T}_n$?
  2. How many nodes with 2 children should we expect in the lemniscate tree of a random polynomial?

Example: Lemniscate trees with 40 vertices

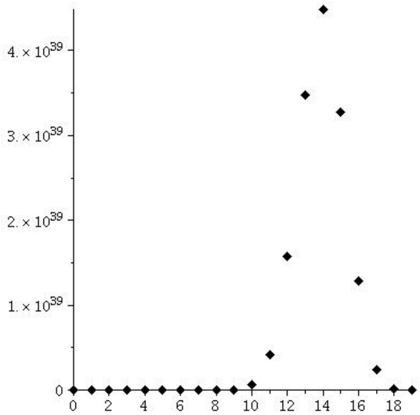

Figure: frequency of $N_2 = k$, $n = 40$

The Generating Function

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- $F(z,u) := \sum_{n,k=0}^{\infty} \frac{a_{n,k}}{n!} u^k z^n$
- They show:

$$F(z,u) = \left[\cosh\left(\frac{z}{2}\sqrt{1-2u}\right) - \frac{\sinh\left(\frac{z}{2}\sqrt{1-2u}\right)}{\sqrt{1-2u}}\right]^{-2}$$

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- So $[z^n]F(z, 1) = \frac{8(n+1)}{\pi^2} \left(\frac{2}{\pi}\right)^n + O(R^{-n})$
- Thus $|\mathcal{T}_n| = n![z^n]F(z, 1) \sim \frac{8}{\pi^2} \left(\frac{2}{\pi}\right)^n (n + 1)!$

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  - For bivariate generating functions (treating one variable as a parameter)
  - The example at hand falls into the "movable singularity schema"

Perturbed Singularity Analysis

- For u in a neighborhood of 1, the nearest singularity of $F(z,u)$ moves, namely to $\rho(u) = \frac{1}{\sqrt{1-2u}} \log\left(\frac{1+\sqrt{1-2u}}{1-\sqrt{1-2u}}\right)$, but it's nature as an order two pole remains the same

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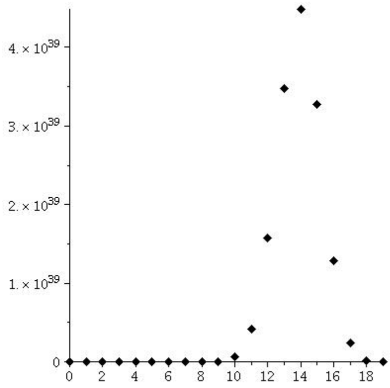
- This ultimately gives rise to a Gaussian limit law:

Distribution of nodes with 2 children in $\mathcal{T}_n$

Theorem 1

The random variable $N_2(T_n)$ (where $T_n \in \mathcal{T}_n$ is chosen uniformly at random) converges in distribution to a Gaussian variable with a speed of convergence that is $O(n^{-1/2})$. The mean and variance are asymptotically $(1 - \frac{2}{\pi})n$ and $(\frac{4}{\pi^2} + \frac{2}{\pi} - 1)n$ respectively.



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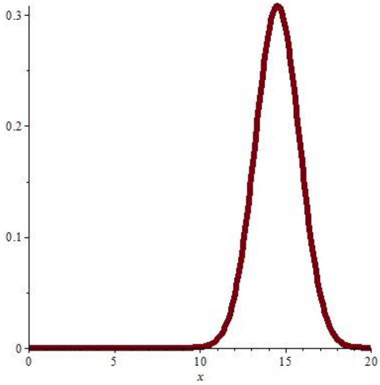

Figure: $N((40 - \frac{80}{\pi}), (\frac{160}{\pi^2} + \frac{80}{\pi} - 40))$

Random Polynomials

In contrast to the previous result, computer simulations show that for different models of random polynomials, such as the Kostlan ensemble, the corresponding lemniscate trees have very little branching. In particular, one can prove the following theorem:

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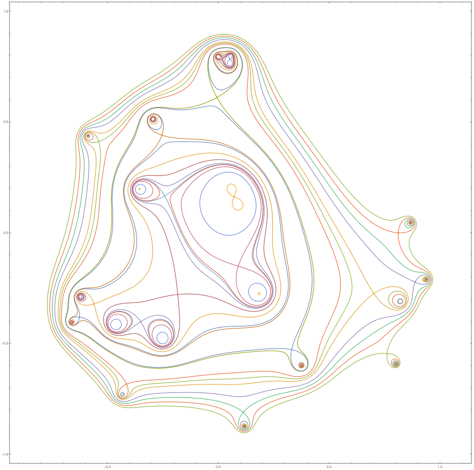
Theorem 2

Let p_N be a random polynomial of degree N whose zeros are drawn i.i.d. from a fixed probability measure μ on S^2 that has a bounded density with respect to the uniform (Haar) measure. Then for every $\epsilon > 0$ there exists C_ϵ so that the number Y_N of nodes with two children in the lemniscate tree associated to p_N satisfies

$$\mathbb{E} Y_N \leq C_\epsilon N^{\frac{1}{2}+\epsilon}.$$

Small Petal Phenomenon



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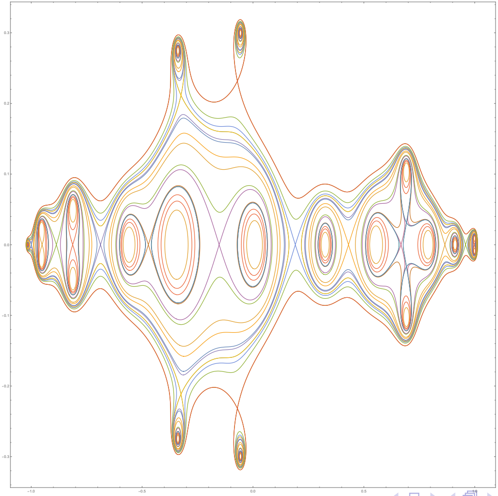
- Random linear combinations of Chebyshev polynomials: $p(z) = \sum_{k=0}^{n} a_k T_k(z)$, $a_k \sim N(0,1)$.
  - rich nesting structure
  - not generally generic

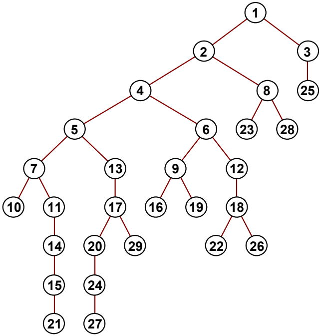
Random Linear Combination of Chebyshev Polynomials



Random Polynomials

Instead we consider a random perturabtion of a Chebyshev polynomial: $p(z) = T_n(z) + \frac{1}{n}\sum_{k=0}^{n-1} b_k T_k(z)$, where the b_k are independently chosen to be 1 or -1 with equal probability.

Randomly Perturbed Chebyshev Polynomials



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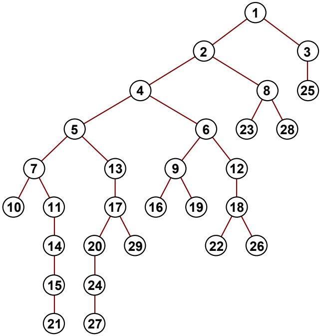

Computer experiments: the lemniscate tree has $\approx n/3$ vertices with two children.

Morse Functions

Let M be a differentiable manifold, $f : M \to \mathbb{R}$ be a smooth function, and $x \in M$.

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- f is a *Morse function* if all of its critical points are nondegenerate.

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- The critical points of any Morse function are isolated.
- If f is a Morse function on a compact differentiable manifold M, then f has at least one (at least 2 if dim $M > 0$) and only finitely many critical points.

Geometric Equivalence

Two Morse functions $f, g : M \to \mathbb{R}$ are said be *geometrically equivalent* if there are orientation preserving diffeomorphisms $R : M \to M$ and $L : \mathbb{R} \to \mathbb{R}$ such that

$$g = L \circ f \circ R.$$

This is an equivalence relation on the set of Morse functions on M.

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The *level function* associated to φ is the function $\ell_\varphi : V(\Gamma) \to \mathbb{Z}$ given by $\ell_\varphi(v) = |\{u \in V(\Gamma) : \varphi(u) \leq \varphi(v)\}|$.

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Connecting Morse functions with Morse Trees

- We can associate to any Morse function on S^2 a Morse tree, and to any Morse tree a Morse function on S^2.

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- We can associate to any Morse function on S^2 a Morse tree, and to any Morse tree a Morse function on S^2.
- Moreover, two Morse functions on S^2 are geometrically equivalent if an only if their associated Morse trees are isomorphic.

The Generating Function

Liviu Nicolaescu established a two-parameter recurrence for enumerating Morse trees, from which he was able to derive a generating function for the number of geometric equivalence classes of Morse functions on S^2. Specifically, he proved

The Generating Function

Theorem (Nicolaescu)

Let $g(n)$ be the number of geometric equivalence classes of Morse functions on the 2-sphere with n saddle points and define

$$\xi(t) = \sum_{n\geq 0} g(n)\frac{t^{2n+1}}{(2n+1)!}$$

Then $\xi(t)$ is the compositional inverse of the function

$$\theta(s) = s\int_0^1 \frac{1}{\sqrt{(1-s^2x^2/2)^2+2s^2x}}\,dx.$$

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The main ingredients in deriving the asymptotic for $g(n)$ are the Lagrange inversion theorem and the saddle point method.

Lagrange Inversion

We can extract the coefficients of $\xi(t)$ from it inverse using the Lagrange inversion theorem:

Lagrange inversion theorem

If ϕ is analytic with $\phi(0) \neq 0$ and $f(w) = \frac{w}{\phi(w)}$, then $[z^n]f^{-1}(z) = \frac{1}{n}[w^{n-1}]\phi(w)^n$.

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This yields $\frac{g(n)}{(2n+1)!} = [t^{2n+1}]\xi(t) = \frac{1}{2n+1}[s^{2n}]\left(\frac{s}{\theta(s)}\right)^{2n+1}$.

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This yields $\frac{g(n)}{(2n+1)!} = [t^{2n+1}]\xi(t) = \frac{1}{2n+1}[s^{2n}]\left(\frac{s}{\theta(s)}\right)^{2n+1}$. Now, finding an asymptotic for $[s^{2n}]\left(\frac{s}{\theta(s)}\right)^{2n+1}$ will allow us to compute the asymtotic for $g(n)$.

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Lagrange inversion theorem

If ϕ is analytic with $\phi(0) \neq 0$ and $f(w) = \frac{w}{\phi(w)}$, then $[z^n]f^{-1}(z) = \frac{1}{n}[w^{n-1}]\phi(w)^n$.

This yields $\frac{g(n)}{(2n+1)!} = [t^{2n+1}]\xi(t) = \frac{1}{2n+1}[s^{2n}]\left(\frac{s}{\theta(s)}\right)^{2n+1}$. Now, finding an asymptotic for $[s^{2n}]\left(\frac{s}{\theta(s)}\right)^{2n+1}$ will allow us to compute the asymtotic for $g(n)$. We can express this as an integral whose asymptotic can be computed using the saddle point method.

The Saddle Point Method

The saddle point method is a method to compute the asymptotics of contour integrals of functions depending on a large parameter, such as Cauchy coefficient integrals

$$[z^n]f(z) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz.$$

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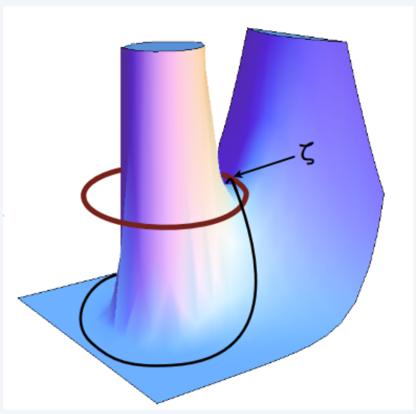
$$\frac{1}{2\pi i} \int_C F(z)\, dz \sim \frac{F(\zeta)}{\sqrt{2\pi f''(\zeta)}}.$$

The Saddle Point Method



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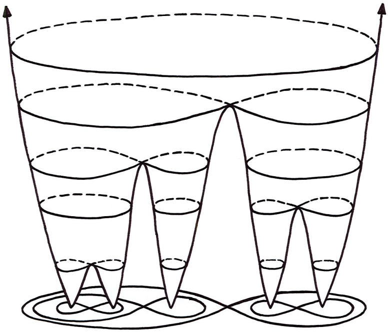
Asymptotic for $g(n)$

Almost Theorem

There are constants $\rho \approx 0.769867$ and $C \approx 1.88056$ such that

$$g(n) = (2n+1)!\rho^n \frac{C}{n^{3/2}}(1+o(1)) \text{ as } n \to \infty.$$

Thank You for Listening!



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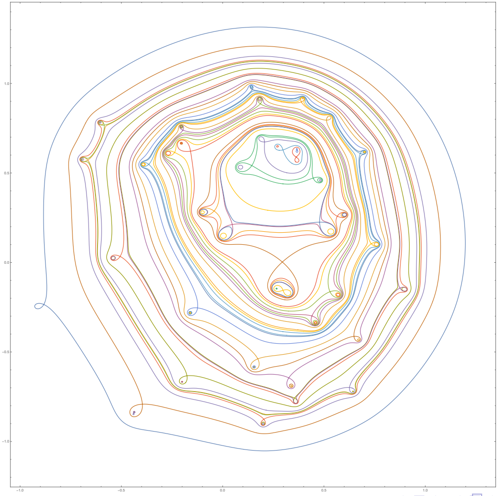

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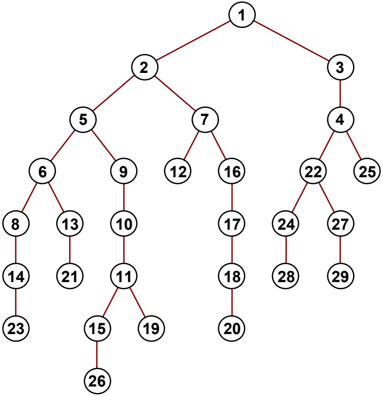

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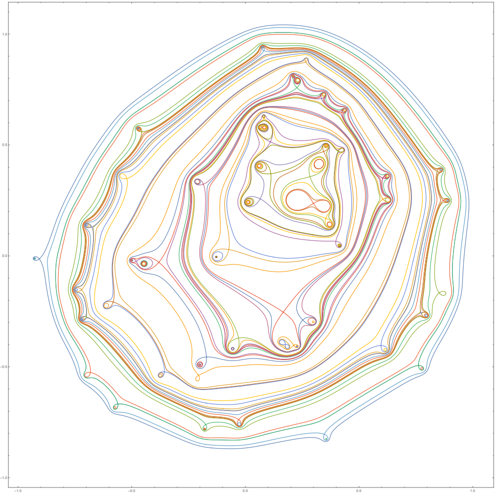

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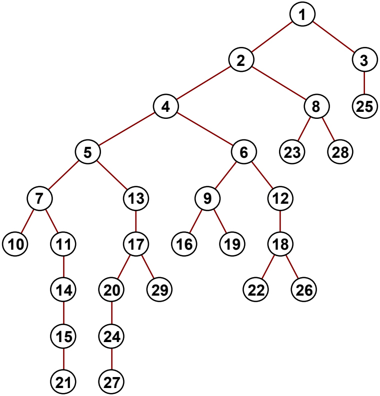

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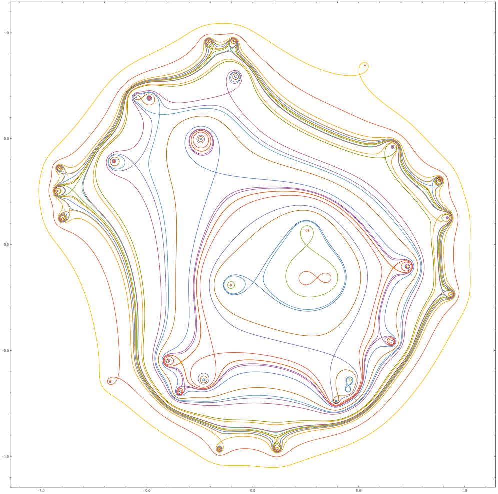

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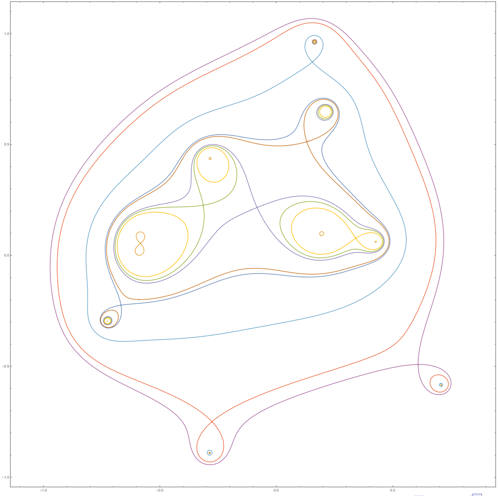

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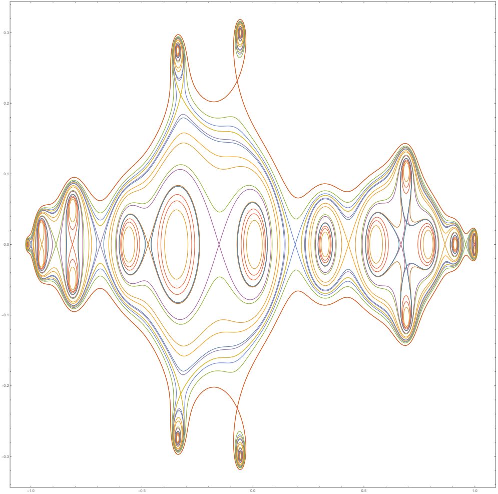

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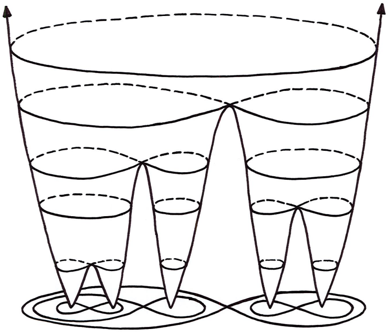

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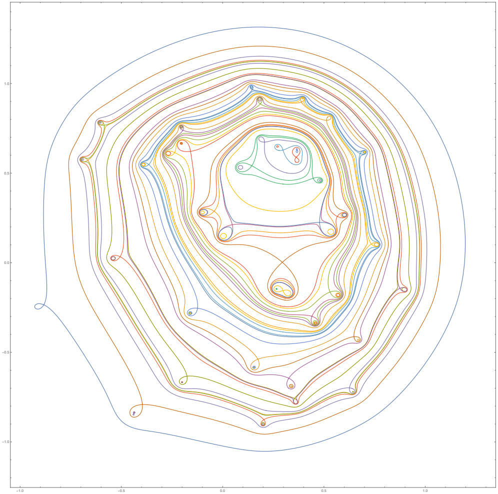

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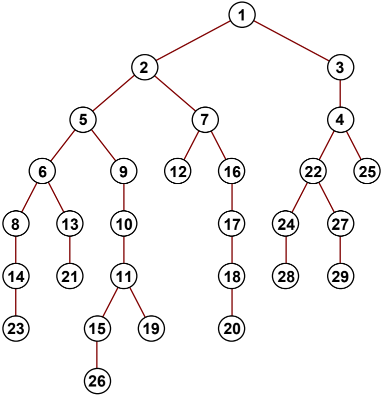

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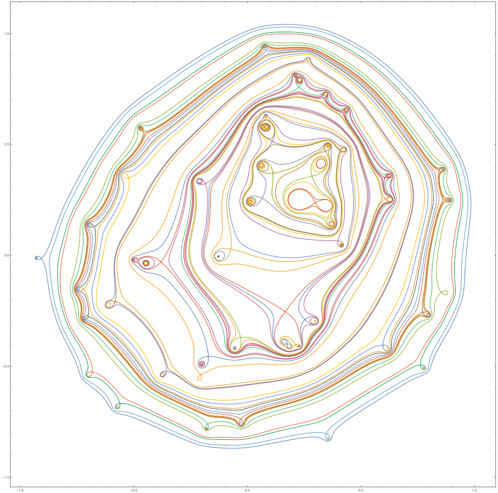

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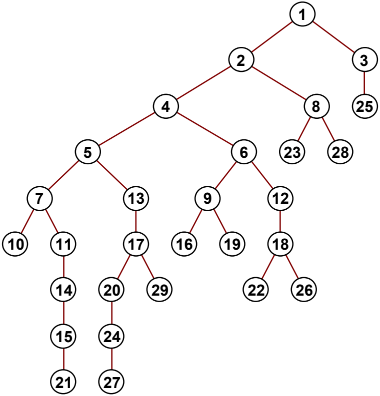

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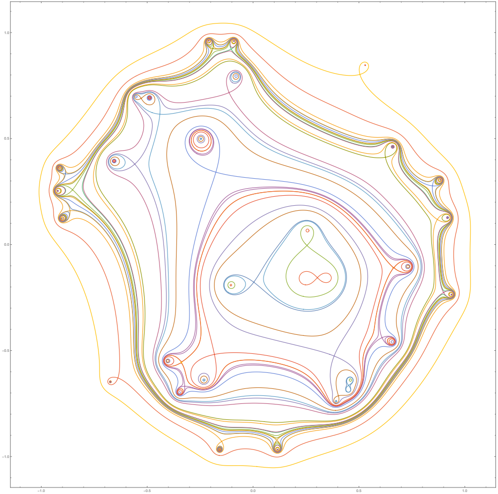

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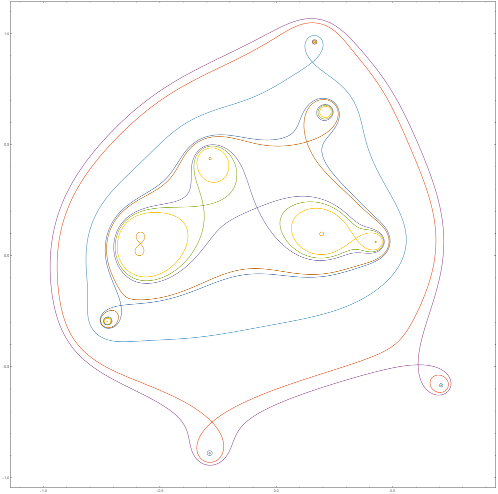

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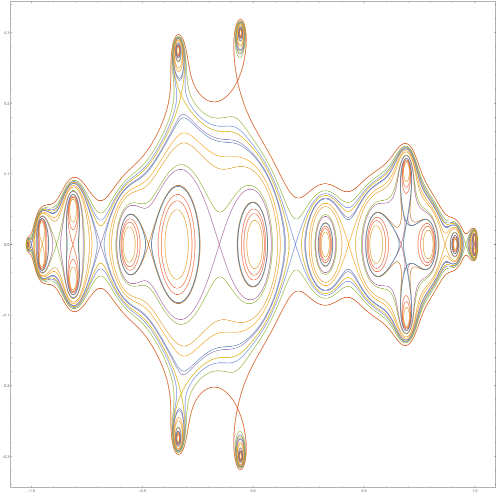

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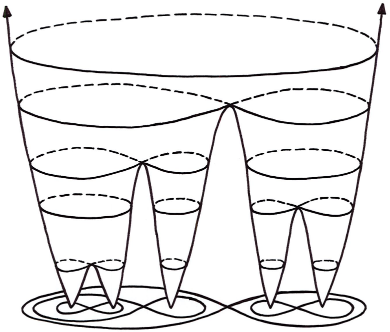

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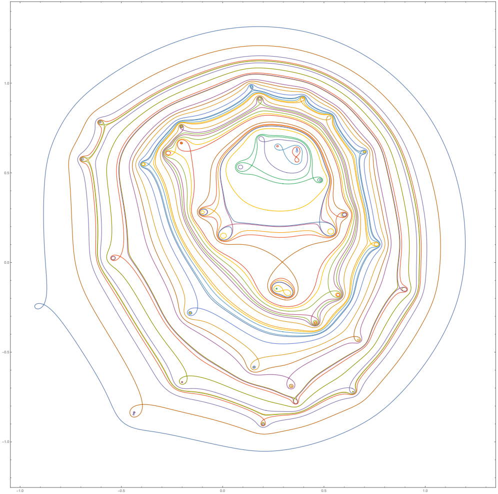

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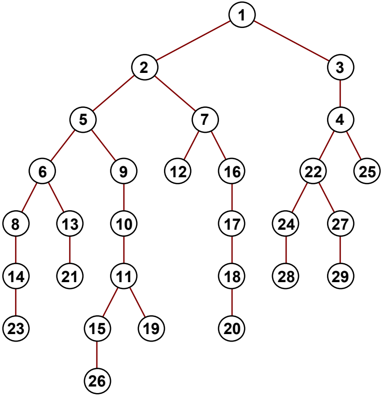

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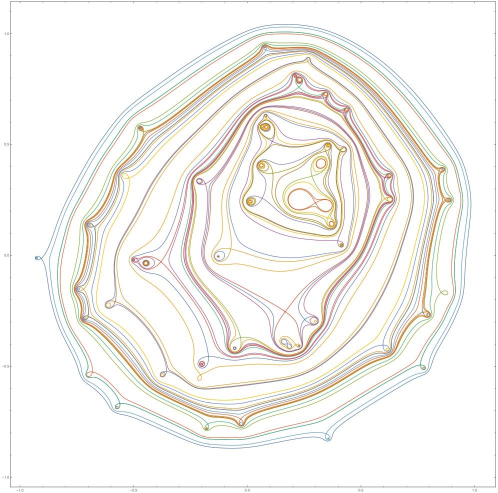

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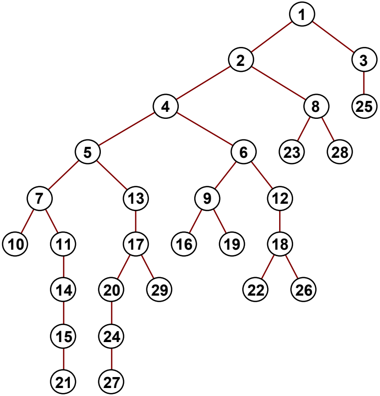

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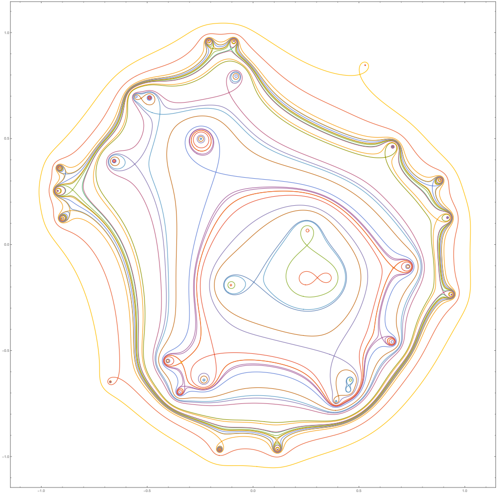

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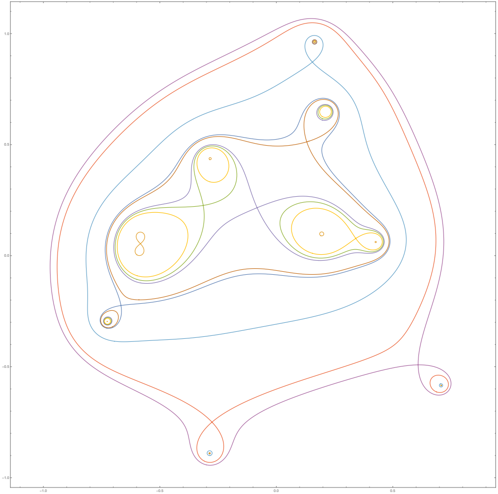

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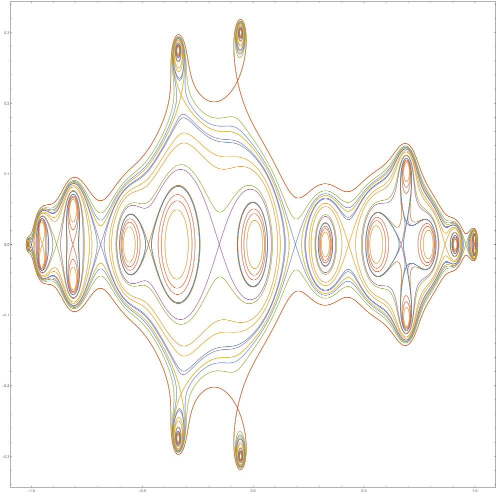

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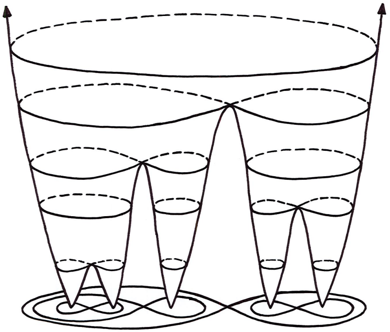

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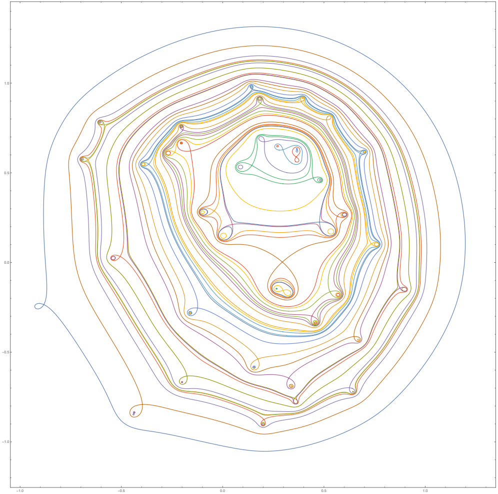

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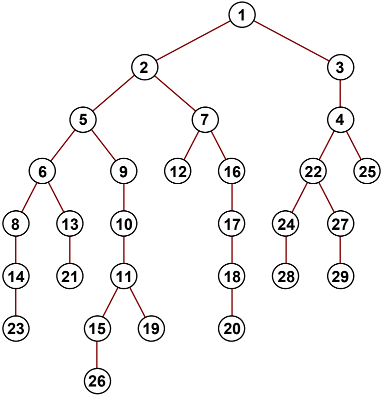

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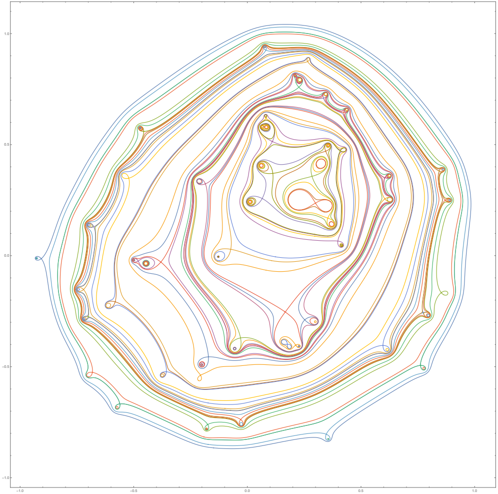

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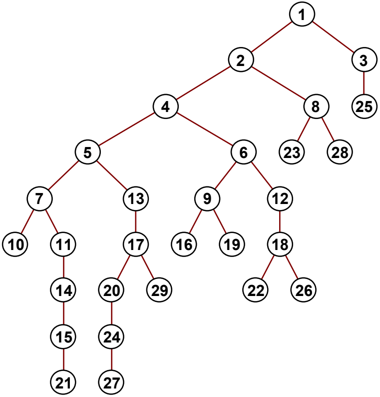

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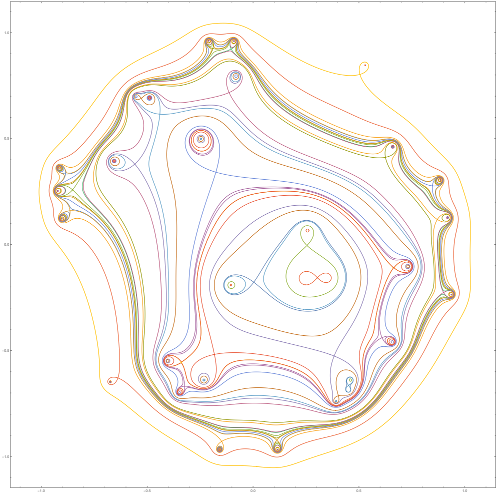

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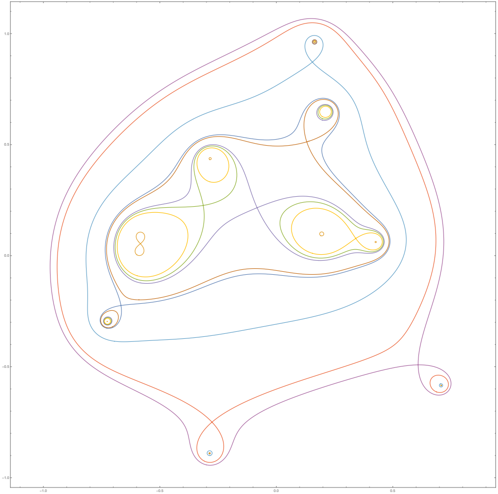

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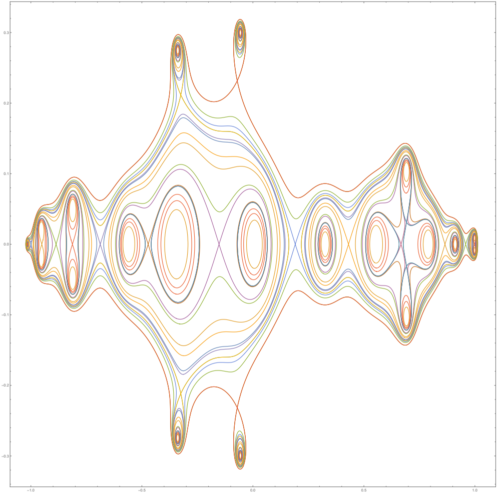

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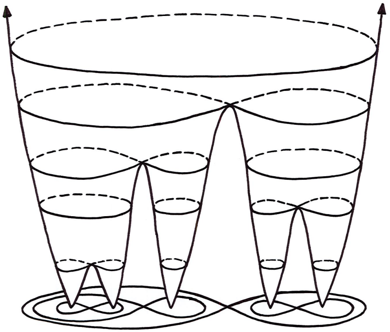

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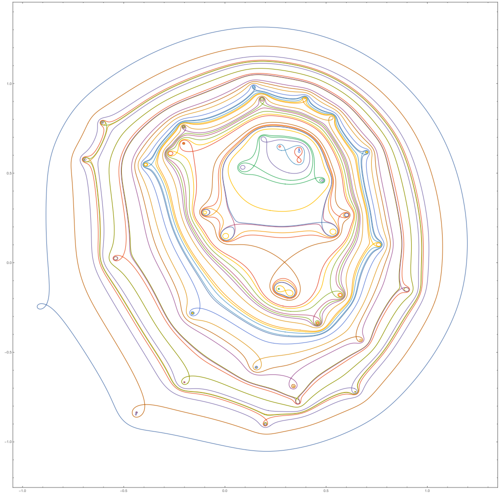

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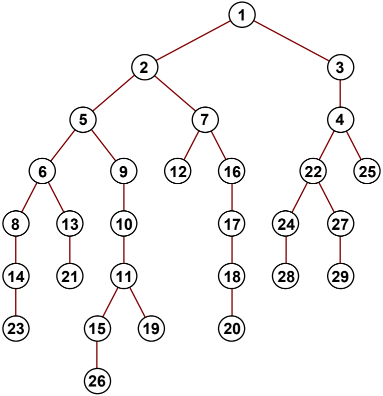

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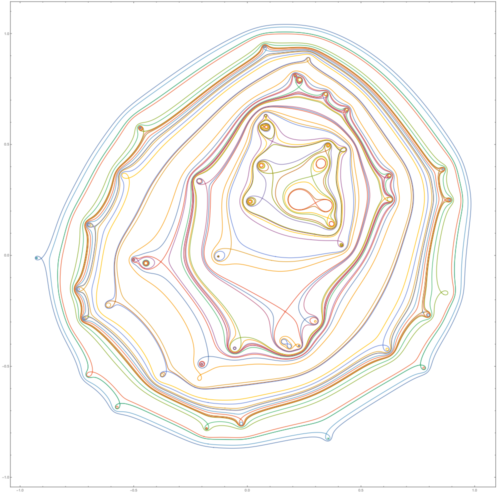

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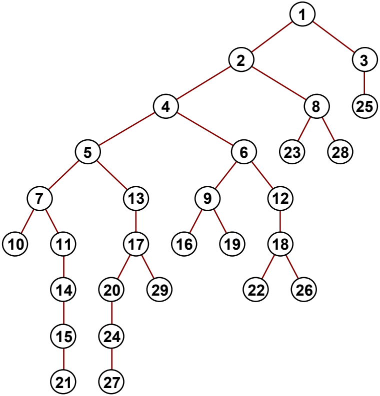

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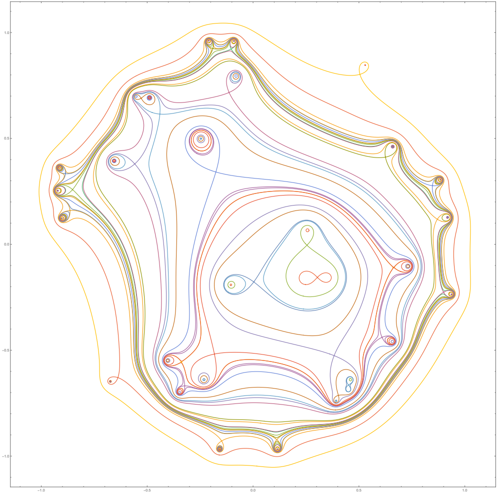

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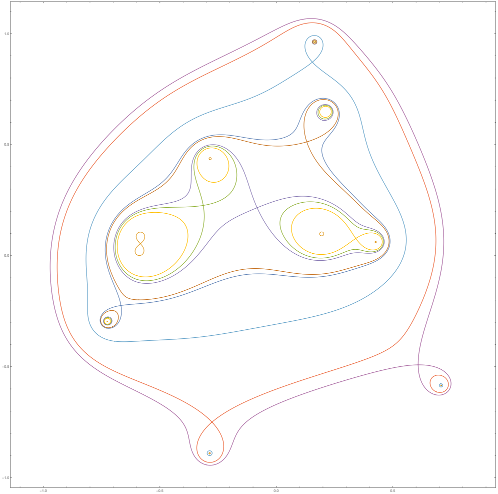

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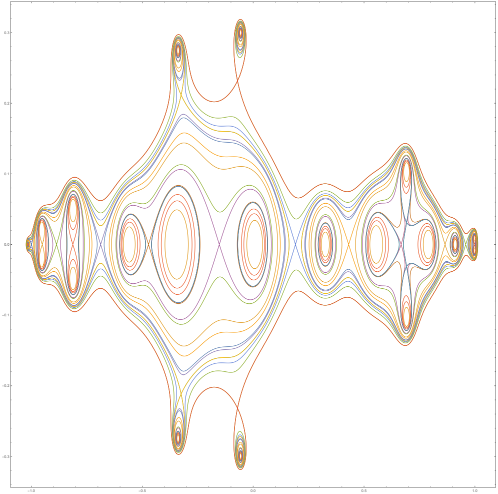

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