Waring's Problem for polynomials

(Content liberally taken from "Waring's Problem, Secant varieties of Veronese Varieties and Parameter spaces for Gorenstein ideals" by Anthony Geramita in The Curves Seminar at Queen's, vol. X)

First: Waring's problem in number theory

1770 Waring conjectured:

For every natural number $j \geq 2$ there is a number $N(j)$ so that every natural number n can be written as

$$n = a_1^j + \cdots + a_{N(j)}^j \qquad [\, a_1, \ldots, a_{N(j)} \geq 0 \,]$$

If $N(j)$ exists, call smallest such integer $\underline{g(j)}$.

Lagrange (1770): Every positive integer is the sum of at most 4 squares $\Longrightarrow$ $g(2) = 4$

Hilbert (1909): $g(j)$ exists for all j

Known: $g(3) = 9$ [every integer can be written as sum of at most 9 cubes]

1986 $\rightsquigarrow$ $g(4) = 19$

$g(5) = 37$

Conjecture: $g(n) = 2^n + \left\lfloor \left(\frac{3}{2}\right)^n \right\rfloor - 2$

"Little" Waring problem: determine $g(n)$

"Big" Waring problem:

$g(3) = 9$ but only $23, 239$ actually require 9 cubes!

Only 15 integers require 8 cubes!

Question: Can 'most' positive integers be written as sum of 7 cubes?

$G(j) :=$ smallest integer so that all sufficiently large integers can be written as a sum of $G(j)$ j^{th} powers.

On to polynomials:

$S = k[x_0, \dots, x_n] \curvearrowright \mathbb{P}^n$

$S_d =$ vector space over k of homogeneous polynomials of degree d

$= \text{span}\{x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} : \sum a_i = d\}$

Remark: $\dim S_d = \binom{n+d}{n}$

Ex: $S = k[x,y]$

$$\binom{1+4}{1} = 5$$

$S_4 = \text{span}\{x^4, x^3y, x^2y^2, xy^3, y^4\}$

$S = k[x,y,z]$

$$\binom{2+2}{2} = 6$$

$S_2 = \text{span}\{x^2, xy, y^2, xz, yz, z^2\}$

Analog of Waring's Problem:

If $F \in S_d$, are there linear forms $l_1, \dots, l_k$ so that

$$F = l_1^d + \dots + l_k^d \, ?$$

Wrk(F)

Smallest such k is called the Waring rank of F.

Ex: $x^3 + y^3$ has Waring rank 2. [except: $(x+y)^3 = x^3 + y^3$, char(k)=3]

Not obvious fact: If $l_1, \dots, l_t$ chosen generically, $l_1^d, \dots, l_t^d$ are linearly independent if $t \le \binom{n+d}{n}$

Cor: If $F \in S_d$, $\mathrm{Wrk}(F) \le \binom{n+d}{n}$. [It exists!]

"Little" Waring problem: What is $\max\{\mathrm{Wrk}(F) : F \in S_d\}$?

Notation: $\boxed{\Phi_{k,d}}$: $\underbrace{S_1 \times \dots \times S_1}_{k} \longrightarrow S_d$

$$(l_1, \dots, l_k) \longrightarrow \sum_{i=1}^{k} l_i^d$$

"Little" Waring problem: Find $\min\{k : \Phi_{k,d} \text{ is surjective}\}$

"Big" Waring problem: Find $\min\{k : \overline{\mathrm{im}(\Phi_{k,d})} = S_d\}$

Ex: Binary Forms of degree 3 → stuff in $k[x,y]$

x^2y has Waring rank 3.

e.g. $(\alpha y)^3 + (\beta(x+y))^3 + (\gamma(x-y))^3$

$x^2y = -\frac{1}{3}y^3 + \frac{1}{6}(x+y)^3 - \frac{1}{6}(x-y)^3$

But general binary form of degree 3 has Waring rank 2

$(ax+by, cx+dy) \to (ax+by)^3 + (cx+dy)^3$

Reason: $\Phi_{2,3} : S_1 \times S_1 \longrightarrow S_3$ ⇝ algebraic, not linear!

$(L_1, L_2) \to L_1^3 + L_2^3$

4 dim. 4 dim. ⇝ $sp\{x^3, x^2y, xy^2, y^3\}$

"Big" Waring problem has a solution! Due to Alexander-Hirschowitz. (dim of secant varieties of Veronese maps)

"Little" Waring problem unsolved in general.

For binary forms of degree d, both solved

Sylvester: General form of degree $2j-1$ in $S=k[x,y]$ has Waring rank j.

$$\underbrace{S_1 \times \cdots \times S_1}_{\dim 2j} \longrightarrow \underbrace{S_{2j-1}}_{\dim 2j}$$

If $F \in S_n$, F is sum of $\leq n$ n^{th} powers.

$x^{n-1}y$ requires n n^{th} powers.

Geometry behind sums of powers

Consider $\tilde{V}_d : S_1 \longrightarrow S_d$ $\qquad (a_0x_0 + \dots + a_nx_n)^d$

$$L \longrightarrow L^d$$

$$(a_0, \dots, a_n) \longrightarrow \left(\cancel{\binom{d}{b_0, \dots, b_n}} \boxed{a_0^{b_0} \cdots a_n^{b_n}} : b_0 + \dots + b_n = d \right)$$

Similar map $V_d : S_1 \longrightarrow S_d$

$$(a_0, \dots, a_n) \longrightarrow \left(a_0^{b_0} \cdots a_n^{b_n} : b_0 + \dots + b_n = d \right)$$

$\operatorname{im}(V_d)$ is called the Veronese Variety

Ex: $\boxed{S = k[x,y]}$

$V_3 : S_1 \longrightarrow S_3$ $\qquad \boxed{\operatorname{Sec}_2(V_3(S_1))}$

$(a,b) \longrightarrow (a^3, a^2b, ab^2, b^3)$

$(0, 1, 0, 0)$

Upshot: Veronese variety "is" forms of degree d which are powers of linear forms.

$\operatorname{Sec}_k(V_d(S_1))$

$F \in S_d \rightsquigarrow F = L_1^d + \dots + L_k^d \Leftrightarrow$ "F" is on plane spanned by $L_1^d, \dots, L_k^d \rightsquigarrow (k-1)$-dim'l secant plane of $V_d(S_1)$.

Back to binary cubics

$S = k[x,y]$, $F = ax^3 + bx^2y + cxy^2 + dy^3$ "in $\mathbb{P}^3$"

$V_3(a,b) = (a^3, a^2b, ab^2, b^3)$. F on tangent line to $V_3(S_1)$

$\Rightarrow \operatorname{wrk}(F) = 3!$

Tensor rank and quadratic forms

- Homogeneous poly F in S_d is a supersymmetric tensor.
- $L \in S_1 \Rightarrow L^d$ is a rank 1 supersymmetric tensor.
- $\mathrm{Wrk}(F)$ = "supersymmetric" tensor rank of F
  (can be different from tensor rank of F)

Waring rank of quadratic forms

$$\underline{x} = (x_0, \dots, x_n) = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix}$$

$$Q(\underline{x}) = \sum_{0 \le i \le j \le n} a_{ij} x_i x_j$$

$$= \underline{x}^T \overbrace{\begin{bmatrix} a_{11} & \frac{1}{2}a_{12} & \cdots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{21} & & & \vdots \\ \vdots & & & \\ \frac{1}{2}a_{n1} & \cdots & & a_{nn} \end{bmatrix}}^{A} \underline{x}$$

$$D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$$

$$= \underline{x}^T \underbrace{O^T D (O}\, \underline{x})$$

$$= (O\underline{x})^T D (O\underline{x})$$

$$= \sum d_{ii} (O\underline{x})_i^2$$

Ex: $x^2 + xy + y^2 = [x\ y] \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$= [x\ y] \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boxed{\frac{1}{2}\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)^2 + \frac{3}{2}\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}$$

<u>Upshot</u>: Waring rank of a quadratic form
= rank of its corresponding symmetric matrix

Solution to "Big" and "little" Waring problems
is # variables.

(Every quadratic form in $n+1$ variables can
be written as a sum of $\leq n+1$ squares
of linear forms) ↖ 'Little'